Information Retrieval

WS 2012 / 2013

Lecture 4, Wednesday November 14th, 2012

(Compression and Entropy)

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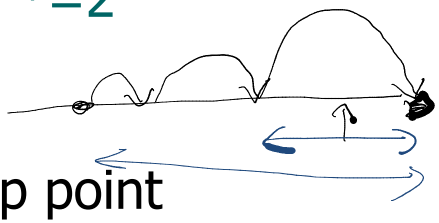
Overview of this lecture

- Organizational
  - Your results and experiences with ES#3 (List Intersection)
- Compression
  - Important to save (index) space
  - But also to save query time !
  - We will see some compression schemes relevant for IR
  - Analyze entropy = information content
  - Shannon's source coding theorem / optimal codes
  - **Exercise Sheet 4:**

    Prove a variety of interesting properties about coding schemes and their entropy

Experiences with ES#3 (list intersection)

- Summary / excerpts   last checked November 14, 16:08
  - Confusion about random list generation ... sorry!
  - Stuff about code optimization was very interesting
  - Math.pow in java is *really* slow ... just use gap *=2
  - Some of the Java people had space problems
  - Binary search can start from last exp-search jump point
  - Some were excited about their code improvements
  - Others were frustrated that they didn't achieve more
  - Live coding was nice again / unit test was helpful
  - Thanks again for great tutor feedback
  - How to autocomplete in Vim ... see my .vimrc on the Wiki



Results for ES#3 (list intersection)

- Main observations + discussion
  - For R=5 hard to make ExpBin faster

    most: ExpBin a bit slower; few: much slower; few: faster
  - For R=50 ExpBin usually faster

    most: somewhat faster; few: a lot faster; few: slower
  - **Reason:** ExpBin algorithmically better, but more complex code (larger constant factor + harder to optimize m.code)
  - Also Simple is faster for R=50 than for R=5
  - **Reason:** long runs can be skipped in single while loop
  - Some Java codes were faster than the fastest C++ codes
  - **Reason:** are you sure your code is correct?

Compression

- Motivation
  - Index lists can be very large for large text collections
  - May have to be stored on disk
  - Compression then obviously saves space
  - But also query time:

    Reading an inverted list from disk takes time

    Typical disk read rate: 50 – 100 MB / second
  - Assume 50 MB / sec and an inverted list of size 50 MB

    Then reading that list from disk takes 1 second

    If we compress it to 10 MB, reading takes 0.2 second

    We need to decompress it then, but even if that takes 0.3 seconds, we have still gained a factor of two !

Compressing inverted lists

- Example of an inverted list of document ids

  3, 17, 21, 24, 34, 38, 45, ..., 11876, 11899, 11913, ...

  - Numbers can become very large ... so we need 4 bytes to store each, for web search even more
  - But we can also store the list like this

  +3, +14, +4, +3, +10, +4, +7, ..., +12, +23, +14, ...

  - This is called **gap encoding**
  - Works as long as we process the lists **from left to right**
  - Now we have a sequence of mostly small numbers
  - We need a scheme to store small numbers in few bits

Universal encoding

- Encode small (positive) integers in few bits
  - **Ideally:** use $\log_2 x$ bits to encode $x \in N$
  - We certainly can't do better than that

    $\log_2 n$ bits needed to differentiate between n numbers
  - We can't even get $\leq 2 \cdot \log_2 x$ bits ... see Exercise 4.3 !

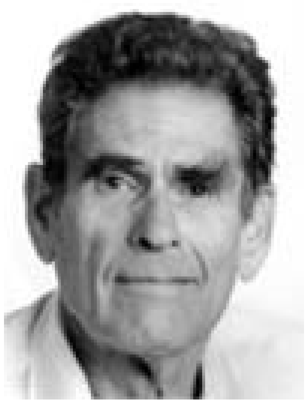
Prefix-free codes

- For our purposes, codes should be **prefix-free**
  - That is: no encoding of a symbol must be a prefix of an encoding of some other symbol
  - Assume the following code (which is not prefix-free)
    - A encoded by 1, B encoded by 11
    - now what does the sequence 1111 encode?
    - could be AAAA or ABA or BAA or AAB or BB
  - For a prefix-free code, decoding is unambiguous
  - And so are all the codes we will consider in this lecture

Elias encodings 1/2

- Elias-Gamma encoding, from 1975
  - Write x in binary, and prepend floor($\log_2 x$) zeros
  - Prefix-free, intuitively because the initial zeros tell us how long the binary representation of x is ... Exercise 4.1
  - Code for x uses ≈ 2 $\log_2 x$ bits ... exact length: Exercise 4.2
  - Let's look at the Elias-Gamma codes of 1, 2, 3, 4, 5, ...

1	1
010	2
011	3
00100	4
00101	5
00110	6



*1923 New Jersey
†2001 Massachusetts

Elias encodings 2/2

- Elias-Delta encoding, also from 1975
  - Write x in binary, prepend Elias-Gamma code of $\text{floor}(\log_2 x) + 1$
  - Prefix-free for basically the same reason ... Exercise 4.1
  - This requires $\log_2 x + O(\log \log x)$ bits
  - Let's look at the Elias-Delta codes of 1, 2, 3, 4, 5, ...

1 1	1
0 1 0 1 0	2
0 1 0 1 1	3
0 1 1 1 0 0	4
0 1 1 1 0 1	5
0 1 1 1 1 0	6

Entropy encoding

- What if the numbers are not in sorted order
  - Or not numbers at all but just symbols

    C C B A D B B A B B C B B C B D

    0101100010010 . . . .
  - Give each number a code corresponding to its **frequency**
  - Frequencies in our example: A: 2 B: 8 C: 4 D: 2
  - A prefix-free code: B → 1 C → 01 D → 0010 A → 0001

    Requires: $8 \cdot 1 + 4 \cdot 2 + 2 \cdot 4 + 2 \cdot 4 = 32$ bits

    That is: 2 bits / symbol on average

    Better than the obvious 3-bit code
  - How do we know if / when we have reached the optimum?

Entropy 1/7

$H(x) = \sum_i p_i \log_2 \frac{1}{p_i}$

$p_i \leq 1 \Rightarrow H(x) \geq 0$

- Definition
  - **Intuitively:** the information content of a message = the optimal number of bits to encode that message
  - **Formally:** defined for a discrete random variable X
  - Without loss of generality range of X = {1, ..., m}

    Think of X as generating the symbols of the message
  - Then the **entropy** of X is written and defined as

    $H(X) = - \Sigma_i p_i \log_2 p_i$ where $p_i = \text{Prob}(X = i)$

$x \log_2 x \xrightarrow[x \to 0]{} 0$

EXAMPLE 1: $p_i = \frac{1}{m} \; \forall i$ $H(x) = \log_2 m$ equi-distr.

EXAMPLE 2: $p_j \Rightarrow 1$ for one j $H(x) = 0$

$p_i \to 0$ for other i

Entropy 2/7

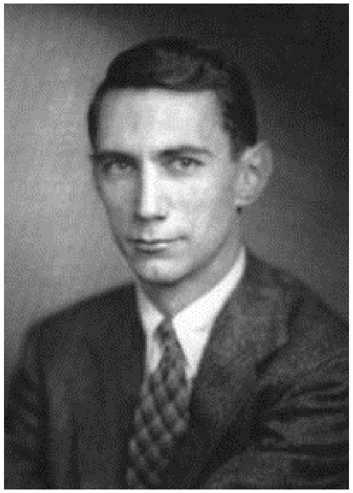
- Shannon's famous source coding theorem (1948)
  - Let X be a random variable with finite range
  - For an arbitrary prefix-free (**PF**) encoding C let $L_C(x)$ be the length of the code for $x \in \text{range}(X)$

    (1) For any PF encoding C it holds: $\mathbf{E}\, L_C(X) \geq H(X)$

    (2) There is a PF encoding C with: $\mathbf{E}\, L_C(X) \leq H(X) + 1$

    where $\mathbf{E}$ denotes the expectation
  - **Intuitively:** no code can be better than the entropy, and there always is a code which is almost as good

*1916 Michigan
†2001 Massachusetts



Entropy 3/7

- Proof of the source coding theorem
  - Denote by Li the length of the code for the i-th symbol
  - The following lemma is key for the source coding theorem:

    (1) Given a PF code with lengths Li ⇨ $\Sigma_i 2^{-Li} \leq 1$

    (2) Given Li with $\Sigma_i 2^{-Li} \leq 1$ ⇨ exists PF code with length Li

    $\Sigma_i 2^{-Li} \leq 1$ is known as "Kraft's inequality"

Entropy 4/7

1 ←

1 1 1

- Proof of Lemma, part (1)

  Given a PF code with lengths Li ⇨ $\Sigma_i\, 2^{-Li} \leq 1$

Consider the following random experiment:

101101...

0 or 1 with prob. $\frac{1}{2}$ until I get a code OR no more code possible

This is well-defined! (only for PF codes)

C_i = the event that I get code i.

$$(*) = \Pr(C_1 \text{ or} \ldots \text{ or } C_m)$$

$$= \sum_{i=1}^{m} \Pr(C_i) = \sum_{i=1}^{m} 2^{-L_i} \leq 1$$

↑ because (*) is a probability

Entropy 5/7

- Proof of Lemma, part (2)

  Given Li with $\sum_i 2^{-L_i} \leq 1$ ⇨ exists PF code with length Li

For example $\underset{L_1}{3}, \underset{L_2}{3}, \underset{L_3}{1}, \underset{L_4}{2}$

Check that

$\sum 2^{-L_i} = 1$

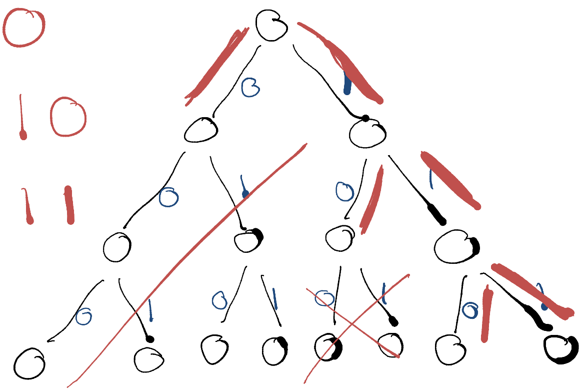
$\underset{1/8}{2^{-3}} + \underset{1/8}{2^{-3}} + \underset{1/2}{2^{-1}} + \underset{1/4}{2^{-2}}$

$C_3 = 0$

$C_4 = 10$

$C_2 = 110$

$C_1 = 111$



This is the principle behind Huffman Encoding

Entropy 6/7

Correction 26 Nov 12 20:47
* tidier write-up
* use equality for side constraint, by introducing variable s

- Proof of source coding theorem, part (1)

For any PF encoding C it holds: **E** $L_C(X) \geq H(X)$

By def. of expectation: $E\,L(X) = \sum_{i=1}^{m} p_i \cdot L_i$

By Kraft's inequality: $\sum_{i=1}^{m} 2^{-L_i} = s \leq 1$

LAGRANGE AGAIN:

$$\mathcal{L} = \sum_{i=1}^{m} p_i \cdot L_i + \lambda \left(s - \sum_{i=1}^{m} 2^{-L_i} \right)$$

$$\frac{\partial \mathcal{L}}{\partial L_j} = p_j - \lambda \cdot \ln 2 \cdot 2^{-L_j} \overset{!}{=} 0$$

$$\Longrightarrow p_j = \lambda \cdot \ln 2 \cdot 2^{-L_j}$$

$$\Longrightarrow 1 = \sum_{i=1}^{m} p_i = \lambda \cdot \ln 2 \cdot \sum_{i=1}^{m} 2^{-L_i} \Rightarrow \lambda \cdot \ln 2 = 1/s$$

$$\Longrightarrow p_j = \underbrace{1/s}_{\geq 1} \cdot 2^{-L_j} \Longrightarrow L_j \geq \log_2 \frac{1}{p_j}$$

$$\frac{\partial^2 \mathcal{L}}{\partial^2 L_j} > 0 \Rightarrow \text{MIN} \Rightarrow \sum_{i=1}^{m} p_i \cdot L_i \geq \sum_{i=1}^{m} p_i \cdot \log_2 \frac{1}{p_i} = H(X) \blacksquare$$

$$[e^x]' = e^x$$

$$2^x = e^{\ln 2 \cdot x}$$

$$\Rightarrow [2^{-x}]' = -\ln 2 \cdot e^{-\ln 2 \cdot x} = -\ln 2 \cdot 2^{-x}$$

Entropy 7/7

$$H(X) = \sum p_i \cdot \log_2 \frac{1}{p_i}$$

- Proof of source coding theorem, part (2)

  There is a PF encoding C with: **E** $L_C(X) \le H(X) + 1$

$$E\,L(X) = \sum_{i=1}^{m} p_i L_i \le \sum_{i=1}^{m} p_i \left(\log_2 \frac{1}{p_i} + 1\right)$$

Just set $L_i = \lceil \log_2 \frac{1}{p_i} \rceil \Rightarrow \sum_{i=1}^{m} 2^{-L_i} \le \sum_{i=1}^{m} 2^{-\log_2 \frac{1}{p_i}}$

$= \sum_{i=1}^{m} p_i = 1$

Kraft's inequality

$\Rightarrow \exists$ PF code

$$\Rightarrow = \underbrace{\sum_{i=1}^{m} p_i \log_2 \frac{1}{p_i}}_{H(X)} + \underbrace{\sum_{i=1}^{m} p_i}_{=1} = H(X) + 1 \quad \blacksquare$$

Optimality of Elias-Gamma

- Elias encoding
  - Elias code lengths satisfy $Li \leq 2 \log_2 i + 1$
  - Let $p_i = 1 / i^2$ for $i \geq 2$, and p_1 such that $\Sigma_i\ p_i = 1$

    That is, numbers $i \geq 2$ occur with probability $1 / i^2$
  - Recall **E** $L(X) = \Sigma_i\ p_i\ Li$ and $H(X) = -\Sigma_i\ p_i \log_2 p_i$
  - Then we have **E** $L(X) \leq H(X) + 1$

Golomb encoding 1/2

- A slightly more involved encoding from 1966
  - Comes with a parameter M, called modulus
  - Write positive integer x as q · M + r
  - Where q = x div M and r = x mod M
  - The code for x is then the concatenation of:

    (1) the quotient q written in unary with 0s

    (2) a single 1 (as a delimiter)

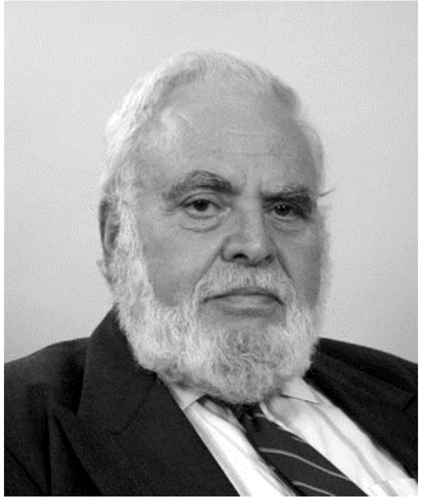
    (3) the remainder r written in binary in $\lceil \log_2 M \rceil$ bits

$M = 10$, $x = 37$

$q = 3$; $r = 7$

code : $000.1.\underbrace{0111}_{\lceil \log_2 M \rceil \text{ bits}}$

Solomon Golomb
*1932 Maryland



Golomb encoding 1/2

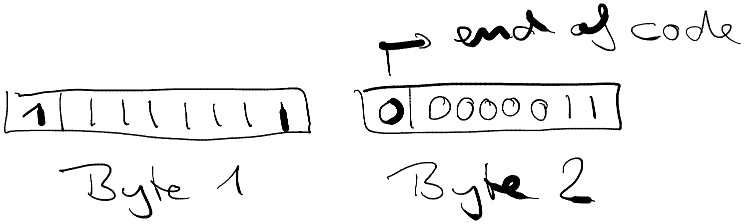
- Analysis
  - Golomb codes are optimal for gap-encoding inverted list

    You should prove this yourself in Exercise 4.4
  - Typically not used in practice, however !
  - **Reason:** the additional decompression effort usually does not outweigh the slight improvement in space, compared to simpler schemes

Variable-Byte Encoding

- A very simple scheme often used in practice
  - Use **whole bytes**, in order to avoid the (computationally expensive) bit fiddling needed for the previous schemes
  - Use one bit of each byte to indicate, whether this is the last byte in the current code or not
  - This is also used for the UTF-8 encoding ... later lecture

511 = 111111111



References

- In the Raghavan/Manning/Schütze textbook

  Section 5: Index compression

  Section 5.3: Postings file compression ... (some codes only)

- Relevant Wikipedia articles

  http://en.wikipedia.org/wiki/Elias_gamma_coding

  http://en.wikipedia.org/wiki/Elias_delta_coding

  http://en.wikipedia.org/wiki/Golomb_coding

  http://en.wikipedia.org/wiki/Variable-width_encoding

  http://en.wikipedia.org/wiki/Source_coding_theorem

  http://en.wikipedia.org/wiki/Kraft_inequality