Efficient Route Planning

SS 2011

Lecture 3, Friday May 20th, 2011

(A* with landmarks, correctness proofs)

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Overview of this lecture

- Announcement
  - No lecture next week!
- Feedback from the exercises
  - Your experimental results
  - Your experiences
- A new algorithm
  - Landmarks: a better heuristic for A*
  - How to select good landmarks?
  - Correctness proofs
- Exercises
  - Implement landmark A*
  - Extend proofs + check preconditions

Announcement

- There is no lecture next Friday!
  - The next lecture is on **Friday, June 3**
    (same time, same place)

Experimental results from Ex. Sheet 1

- See the table on the Wiki
  - Many results still missing, please put them there!
  - The results which are there are quite conclusive:
    - Plain Dijkstra on Ba-Wü around 0.5 seconds
    - 20% of all nodes settled on average (= a lot)
    - A* with the straight-line heuristic is at best twice faster
    - A single iteration takes around 0.5 μs

      ... depending on the priority queue implementation

Your experiences with Ex. Sheet 1

- You didn't write much in the SVN
  - ... so please tell me about it now

A* with landmarks 1/3

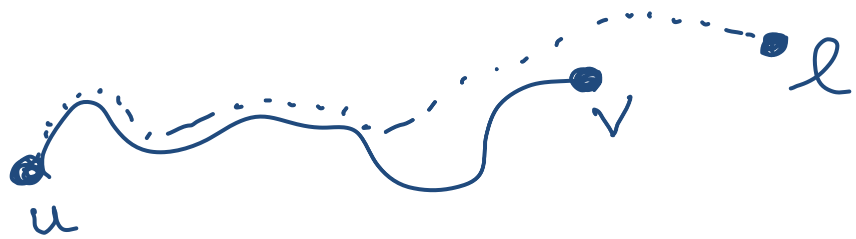
- Basic idea
  - Consider an arbitrary node ℓ and call it a landmark
  - Then for two arbitrary nodes u, v it holds:

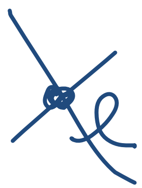
    $\text{dist}(u, \ell) \leq \text{dist}(u, v) + \text{dist}(v, \ell)$ "triangle inequality"

    hence $\text{dist}(u, \ell) - \text{dist}(v, \ell) \leq \text{dist}(u, v)$
  - When is the left hand side a good lower bound?
  - That is, when is $\text{dist}(u, \ell)$ close to $\text{dist}(u, v) + \text{dist}(v, \ell)$?

A* with landmarks   2/3

- When is dist(u, ℓ) close to dist(u, v) + dist(v, ℓ) ?
  - When v lies "close to" the shortest path from u to ℓ
    - Note: if it lies **on** the shortest path we have equality!
  - This is likely if
    - v lies close to the straight line between u and ℓ
    - ℓ is not too far from v (and so, in fact, behind v)
  - Obviously we can't have this for all nodes u and v





A* with landmarks 3/3

- Pick a set L of landmarks
  - For each $\ell \in L$ we have $\mathrm{dist}(u, \ell) - \mathrm{dist}(v, \ell) \leq \mathrm{dist}(u, v)$
  - Hence also $\max_{\ell \in L} \{\mathrm{dist}(u, \ell) - \mathrm{dist}(v, \ell)\} \leq \mathrm{dist}(u, v)$
  - When is the left hand side a good lower bound?
    - The more landmarks the better
    - But to make good use of the lower bound above, we need to **precompute (and store)** distances from each landmark to all other nodes in the graph
    - For a given number of landmarks, the more "distributed" they are over the graph, the better

Landmark selection

- We look at two heuristics
  - Random selection
    - not bad, but will not give perfect distribution
  - Greedy farthest node selection
    - start with a random node, then iteratively add more
    - in each iteration, pick the node that is **farthest** from the set of nodes already selected
      - let the already selected set be L'
      - then pick node u which maximizes $\min_{\ell \in L'} \mathrm{dist}(\ell, u)$
      - how do we pick that node?

Dijkstra from a **set** of nodes

- Implementation
  - Initially put all nodes from the set S in the priority queue, with distance 0, then run ordinary Dijkstra
  - Then the distance computed for each node u will be

    $\min_{s \in S} \text{dist}(s, u)$ ... which we write as $\text{dist}(S, u)$
  - It's not obvious that this is true, so we should prove it
    - This will be one of the exercises
    - Extension of correctness proof for ordinary Dijkstra
      - which I will hence show you again now

Basic Dijkstra correctness proof 1/3

- Let s be our source node
  - Let's first make the simplifying assumptions that the $dist(s, u)$ are **distinct** for all nodes u
  - Then we can order the nodes $u_1, u_2, u_3, \ldots$

    such that $dist(s, u_1) < dist(s, u_2) < dist(s, u_3) < \ldots$
  - We want to prove that, at the end of the computation,
    - the tentative distance $dist[u_i]$ for each node u_i satisfies $dist[u_i] = dist(s, u_i)$
  - More specifically, we can show that in the i-th iteration
    - Dijkstra's algorithm settles node u_i
    - and at that point $dist[u_i] = dist(s, u_i)$

Basic Dijkstra correctness proof 2/3

- We show by induction over i
  - that in the i-th iteration, we have dist[u_j] = dist(s, u_j) for all $j \le i$, and node u_i will be settled in that iteration

$i = 1$: $u_1 = s$; $\text{dist}[s] = 0 = \text{dist}(s, s)$ ✓

$i \to i+1$: for $u_1, \ldots, u_i$ the claim holds
what about u_{i+1}?
Let's look at the shortest path from s to u_{i+1}

$$\text{dist}(s, v) = \text{dist}(s, u_{i+1}) - \underbrace{c(v, u_{i+1})}_{>0} < \text{dist}(s, u_{i+1})$$

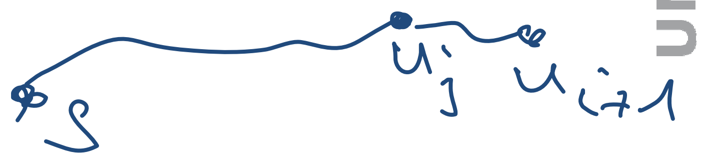
$\Rightarrow$ v is one of the $u_1, \ldots, u_i$

$v = u_j$
$j \le i$

predecessor of u_{i+1} on this shortest path

Basic Dijkstra correctness proof 3/3

$v = u_j \ , \ j \le i$



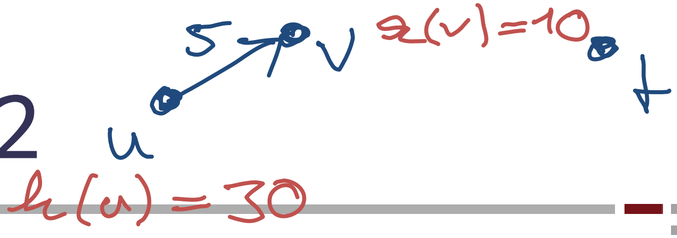
$\Longrightarrow \ \text{dist}[u_j] = \text{dist}(s, u_j).$
and u_j was settled in iteration j

When u_j was settled, $\text{dist}[u_{i+1}]$
received the value $\underbrace{\underbrace{\text{dist}[u_j]}_{= \text{dist}(s, u_j)} + c(u_j, u_{i+1})}_{= \text{dist}(s, u_{i+1})}$

$j > i+1 : \ \text{dist}[u_j] \ge \text{dist}(s, u_j)$
$> \text{dist}(s, u_{i+1}) = \text{dist}[u_{i+1}]$

$\Longrightarrow \ u_{i+1}$ is settled in iteration $i+1$
and $\text{dist}[u_{i+1}] = \text{dist}(s, u_{i+1})$ $\blacksquare$

A* correctness proof 1/2



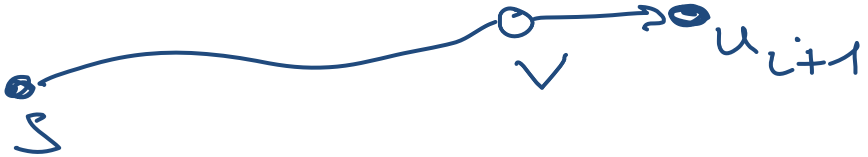
- We make the following assumptions
  - Let t be the target node, and let h(u) be the estimated distance to that target for node u
  - We assume that $h(u) \le dist(u, t)$ "h is admissible"
  - And that for each arc (u, v) with cost c(u, v) it holds that $h(u) \le c(u, v) + h(v)$ "h is monotone"
  - For simplicity, we first assume **strict** monotonicity, that is $h(u) \mathbf{<} c(u, v) + h(v)$

    and that $dist(s, u) + h(u)$ are **distinct** for all nodes u
  - Then we have an ordering of the nodes $u_1, u_2, u_3, \ldots$ with

    $dist(s, u_1) + h(u_1) < dist(s, u_2) + h(u_2) < \ldots.$

A* correctness proof 2/2

$i \to i+1$: for $u_1, \ldots, u_i$ it's correct.



Why is v one of the $u_1, \ldots, u_i$

Want to show: $\text{dist}(s,v) + h(v) < \text{dist}(s,u_{i+1}) + h(u_{i+1})$

$$\text{dist}(s,v) + h(v) \overset{\text{mon. of } h}{\le} \underbrace{\text{dist}(s,v) + c(v,u_{i+1})}_{= \text{dist}(s,u_{i+1})} + h(u_{i+1})$$

$\Rightarrow v = u_i \,, \, i \le j$

$j > i+1$: $\text{dist}[u_j] + h(u_j) \ge \text{dist}(s,u_j) + h(u_j)$

by our ordering of the $u_1, \ldots, u_n$ $\to$ $\ge \text{dist}(s,u_{i+1}) + h(u_{i+1})$

$\Rightarrow$ u_{i+1} is settled in iteration $i+1$
and $\text{dist}[u_{i+1}] = \text{dist}(s,u_{i+1})$ ∎

Admissible and monotone h for A*

- What about the straightline heuristic for h ?
  - It's easy to see that it is admissible and monotone
  - Is it also strictly monotone?
  - This is one of the exercises
- What about the landmark heuristic for h ?
  - Let L be the set of landmarks and t be the target node
  - Then we have $h(u) = \max_{\ell \in L} \{dist(u, \ell) - dist(t, \ell)\}$
  - Admissible: we already showed that $h(u) \leq dist(u, t)$
  - Monotone: we have to show that for all arcs (u,v)

    $h(u) \leq c(u, v) + h(v)$

Monotonicity of landmark heuristic

- Let (u,v) be an arbitrary arc with cost $c(u, v)$
  - We have to show that $h(u) \leq c(u, v) + h(v)$
    - where $h(x) = \max_{\ell \in L} \{dist(x, \ell) - dist(t, \ell)\}$ for all x
  - Let us first show something related for a fixed $\ell \in L$

    $dist(u, \ell) \leq c(u, v) + dist(v, \ell)$ "triangle inequality"

    $\rightarrow$ $dist(u, \ell) - dist(t, \ell) \leq c(u, v) + dist(v, \ell) - dist(t, \ell)$
  - If we now do $\max_{\ell \in L}$ on both sides, we are done
  - But is this ok?

$x_1, \dots, x_{|L|}$ $\quad$ $y_1, \dots, y_{|L|}$

$x_i \leq y_i$

$\max\{x_1, \dots, x_{|L|}\} \leq \max\{y_1, \dots, y_{|L|}\}$

References

- The original landmark paper

  Computing the shortest path: A* search meets graph theory

  A. Goldberg and C. Harrelson, SODA 2005

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